

MULTIDIMENSIONAL COMPOUND POISSON DISTRIBUTIONS IN FREE PROBABILITY

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ABSTRACT. Inspired by R. Speicher's multidimensional free central limit theorem and semicircle families, we prove an infinite dimensional compound Poisson limit theorem in free probability, and define infinite dimensional compound free Poisson distributions in a non-commutative probability space. Infinite dimensional free infinitely divisible distributions are defined and characterized in terms of their free cumulants. It is proved that for a sequence of random variables, the following statements are equivalent. (1) The distribution of the sequence is multidimensional free infinitely divisible. (2) The sequence is the limit in distribution of a sequence of triangular trays of families of random variables. (3) The sequence has a distribution same as that of $\{a_1^{(i)} : i = 1, 2, \dots\}$ of a multidimensional free Levy process $\{a_t^{(i)} : i = 1, 2, \dots : t \geq 0\}$. (4) The sequence is the limit in distribution of a sequence of sequences of random variables having multidimensional compound free Poisson distributions.

Key Words Free Probability, Multidimensional Free Poisson Distributions, Multidimensional free infinitely divisible distributions.

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INTRODUCTION

The most popular distributions in classical probability are Gaussian distributions. Poisson distributions form a class of the most prominent distributions in classical probability beyond Gaussian distributions (Lecture 12 in [NS]). A very important method in classical probability theory for generating distributions centers around a generalization of Poisson distributions. Given a random variable a , one can construct out of this a compound Poisson distribution π_a by the prescription that the moments of a give, up to a common factor, the cumulants of π_a . One possibility to make the transition from a to π_a is by a limit theorem. The importance of these compound Poisson distributions rises from the fact that all infinitely divisible distributions can be approximated by compounded Poisson distributions (see Chapter XVII in [WF] and Section 4.4 in [RS]). These ideas have been imitated in free probability. The counterpart of normal distributions in free probability is semicircle distributions. Such a distribution can be realized as the limit distribution of a sequence of certain random variables. This result was called the *free central limit theorem* proved by D. Voiculescu [DV] (see also Theorem 8.10 in [NS]). Very similarly, a (compound) free Poisson distribution can be realized as the limit in distribution of a sequence of simple distributions (12.11, 12.12, 12.15, and 12.16 in [NS], also [RS], [RS1] and [DV1]). More generally, infinitely divisible distributions, Levy processes, and limit theorems have been studied thoroughly in free probability (see, for instance, [BnT] and [BP]). Very recently, the infinite divisibility of the distribution of a pair of random variables was studied in [GHM] and [MG] in the setting of bi-free probability, a new research area in free probability introduced by Voiculescu in [DV2].

In contrast to the thorough study of free Poisson distributions and free infinitely divisible distributions, we cannot find too much work on multidimensional Poisson distributions and multidimensional infinitely divisible distributions in free probability. Inspired by the corresponding ideas in classical probability, R. Speicher [RS] gave a brief theory on operator-valued compound free

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Poisson distributions and multidimensional free infinitely divisible distributions. Given an m -tuple (a_1, a_2, \dots, a_m) of random variables in a non-commutative probability space (\mathcal{A}, φ) , and a constant number $\lambda \in \mathbb{R}$, Speicher defined a multidimensional compound free Poisson distribution as follows. An m -tuple $\{b_1, b_2, \dots, b_m\}$ of random variables in a non-commutative probability space (\mathcal{B}, ϕ) has a multidimensional compound free Poisson distribution if $\kappa_n(b_{i_1}, b_{i_2}, \dots, b_{i_n}) = \lambda \varphi(a_{i_1} a_{i_2} \dots a_{i_n})$, for $i_1, \dots, i_n \in \{1, 2, \dots, m\}, n \in \mathbb{N}$, where κ_n is the n -th free cumulant on (\mathcal{B}, ϕ) (4.4.1 in [RS]). A multidimensional compound free Poisson limit theorem, the definition of multidimensional infinitely divisible distributions, and the semigroup of distributions and the approximation of compound free Poisson distributions for a multidimensional free infinitely divisible distributions were given in Sections 4.4 and 4.5 in [RS]. All distributions in [RS] are finite dimensional, that is, they are distributions of m -tuples of random variables. Benauch-Georges [BG] characterized an m -dimensional free infinitely divisible distribution in terms of its free cumulants under the hypothesis that all random variables live in a tracial non-commutative probability space (\mathcal{A}, φ) (i. e., $\varphi(ab) = \varphi(ba)$, for all $a, b \in \mathcal{A}$).

R. Speicher [RS1] gave a multidimensional central limit theorem (see also Theorem 8.17 in [NS]). Roughly speaking, the theorem states that the (joint) distribution of a semicircle family can be realized as the limit in distribution of a sequence of families of random variables. Based on the same philosophy, in this paper, we prove an infinite dimensional compound free Poisson limit theorem, and develop a theory on infinite dimensional infinitely divisible distributions in free probability.

This paper is organized as follows. Beside this introduction, there are three sections in the paper. In Section 1, we present some concepts and results in free probability used in sequel. Section 2 is devoted to the study of infinite dimensional compound free Poisson distributions. A free Poisson random variable can be realized as the limit in distribution of a sequence of free triangular trays of scalar multiples of projections (see the arguments at the beginning of Section 2). Substituting the projections in the free Poisson limit theorem by sequences of projections (with possible distinct expectations), we get an infinite dimensional free Poisson limit theorem (Theorem 2.5). By using techniques in ultra-products of C^* -algebras, we can choose the limit sequence of random variables from a C^* -probability space. We say the limit sequence of random variables has an *infinite dimensional free Poisson distribution* (Definition 2.6). By our limit theorem and definition of multidimensional free Poisson distributions, if a sequence of self-adjoint operators in a C^* -probability space has a multidimensional free Poisson distribution, then each random variable in the sequence has a free Poisson distribution. Moreover, a free sequence of free Poisson random variables with certain distributions has a multidimensional free Poisson distribution (Remark 2.7). Using techniques in tensor products of C^* -algebras, we then present an infinite dimensional compound free Poisson limit theorem (Theorem 2.8). We say that the limit sequence of random variables has a *multidimensional compound free Poisson distribution* (Definition 2.10). In Example 2.11, we present examples of sequences of random variables having multidimensional compound free Poisson distributions such as Speicher's multidimensional compound free Poisson distributions given in Section 4.4 of [RS], tuples of random variables constructed from general self-adjoint random variables and a free family of semicircle random variables (a special type of such constructions can be found in Proposition 12.18 in [NS]), and free families of compound free Poisson random variables with certain distributions. Our Example 2.11 (1) shows that the limit theorem and definition of multidimensional compound free Poisson distributions presented in this paper generalize the corresponding theory in Section 4.4 in [RS]. In Section 3, we study infinite dimensional infinitely divisible distributions in free probability. We generalize Speicher's work in Section 4.5 of [RS] and Benauch-Georges' work in [BG] in two aspects. We study distributions of sequences of random variables (we thus call such a distribution an infinite dimensional distribution). Random variables under study are in a C^* -probability space (and its linear functional is not necessarily tracial). We first give the definitions of infinite dimensional infinitely divisible distributions and infinite dimensional free Levy processes $\{\{a_t^{(i)} : i = 1, 2, \dots\} : t \geq 0\}$ (Definitions 3.1 and 3.2). We characterize the free infinite divisibility of the distribution of a sequence of random variables in terms of its free cumulants (Theorem 3.3

(2) and (4)). We also prove that an infinite dimensional distribution is infinitely divisible if and only if it is the distribution of $\{a_1^{(i)} : i = 1, 2, \dots\}$ of an infinite dimensional free Levy process $\{\{a_t^{(i)} : i = 1, 2, \dots\} : t \geq 0\}$ (Theorem 3.3). Finally, using the techniques in inductive limits of C^* -algebras, we prove that a sequence of self-adjoint operators in a C^* -probability space has a multidimensional free infinitely divisible distribution if and only if the distribution of the sequence is the limit distribution of a sequence of infinite dimensional compound free Poisson distributions (Theorem 3.6), which generalize Speicher's free Poisson approximation theorem 4.5.5 in [RS] to the infinitely dimensional distribution case.

1. PRELIMINARIES

In this section we recall some basic concepts and results in free probability used in sequel. The reader is referred to [NS] and [VDN] for more details on free probability, and to [KR] for operator algebras.

Non-commutative Probability spaces. A non-commutative probability space is a pair (\mathcal{A}, φ) consisting of a unital algebra \mathcal{A} and a unital linear functional φ on \mathcal{A} . When \mathcal{A} is a $*$ -unital algebra, φ should be positive, i. e., $\varphi(a^*a) \geq 0, \forall a \in \mathcal{A}$. A C^* -probability space (\mathcal{A}, φ) consists of a unital C^* -algebra and a state φ on \mathcal{A} . A W^* -probability space (\mathcal{A}, φ) consists of a finite von Neumann algebra \mathcal{A} and a faithful normal tracial state φ on \mathcal{A} . An element $a \in \mathcal{A}$ is called a (*non-commutative*) *random variable*. $\varphi(a^n)$ is called the *n-th moment* of a , for $n = 1, 2, \dots$. Let $\mathbb{C}[X]$ be the complex algebra of all polynomials of an indeterminate X . The linear function $\mu_a : \mathbb{C}[X] \rightarrow \mathbb{C}$, $\mu_a(P(X)) = \varphi(P(a)), \forall P \in \mathbb{C}[X]$, is called the *distribution (or law)* of a .

Joint Distributions. Let (\mathcal{A}, φ) be a non-commutative probability space and I be an index set. For a family $\{a_i \in \mathcal{A} : i \in I\}$, the family $\{\varphi(a_{i_1} a_{i_2} \cdots a_{i_n}) : 1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \in I, n \geq 1\}$ is called the family of *joint moments* of $\{a_i : i \in I\}$. Let $\mathbb{C}\langle X_i : i \in I \rangle$ be the unital algebra freely generated by non-commutative indeterminates $X_i, i \in I$. The linear functional $\mu : \mathbb{C}\langle X_i : i \in I \rangle \rightarrow \mathbb{C}$ defined by

$$\mu(P) = \varphi(P(a_{i_1}, a_{i_2}, \dots, a_{i_n})), \forall P = P(X_{i_1}, X_{i_2}, \dots, X_{i_n}) \in \mathbb{C}\langle X_i : i \in I \rangle,$$

is called the *joint distribution* of the sequence $\{a_i : i \in I\}$. A sequence $\{\{a_{i,n} : i \in I\} : n = 1, 2, \dots\}$ of families of random variables in a non-commutative probability space (\mathcal{A}, φ) *converges in distribution* to a family $\{b_i : i \in I\}$ of random variables in a non-commutative probability space (\mathcal{B}, ϕ) if, for all $i_1, i_2, \dots, i_m \in I, m \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} \varphi(a_{i_1,n} a_{i_2,n} \cdots a_{i_m,n}) = \phi(b_{i_1} b_{i_2} \cdots b_{i_m}).$$

Free independence. A family $\{\mathcal{A}_i : i \in I\}$ of unital subalgebras of a non-commutative probability space (\mathcal{A}, φ) is *freely independent (or free)* if $\varphi(a_1 a_2 \cdots a_n) = 0$ whenever the following conditions are met: $a_i \in \mathcal{A}_{l(i)}$, $\varphi(a_i) = 0$ for $i = 1, 2, \dots, n$, and $l(i) \neq l(i+1)$, for $i = 1, 2, \dots, n-1$. A family $\{a_i : i \in I\}$ of elements is free if the unital subalgebras generated by a_i 's are free.

Non-crossing partitions. Given a natural number $m \geq 1$, let $[m] = \{1, 2, \dots, m\}$. A *partition* π of $[m]$ is a collection of non-empty disjoint subsets of $[m]$ such that the union of all subsets in π is $[m]$. A partition $\pi = \{B_1, B_2, \dots, B_r\}$ of $[m]$ is *non-crossing* if one cannot find two block B_i and B_j of π , and four numbers $p_1, p_2 \in B_i, q_1, q_2 \in B_j$ such that $p_1 < q_1 < p_2 < q_2$. The collection of all non-crossing partitions of $[m]$ is denoted by $NC(m)$. $|NC(m)|$, the number of non-crossing partitions of $[m]$, is $C_m = \frac{(2m)!}{m!(m+1)!}$, which is called the *m-th Catalan number* (Notation 2.9 in [NS]).

The Mobius function. Let P be a finite partial ordered set (poset), and $P^{(2)} = \{(\pi, \sigma) : \pi, \sigma \in P, \pi \leq \sigma\}$. For two functions $F, G : P^{(2)} \rightarrow \mathbb{C}$, we define the convolution $F * G$ by

$$F * G(\pi, \sigma) := \sum_{\rho \in P, \pi \leq \rho \leq \sigma} F(\pi, \rho) G(\rho, \sigma).$$

Let $\delta(\pi, \sigma) = 1$, if $\pi = \sigma$; $\delta(\pi, \sigma) = 0$, if $\pi < \sigma$. Then

$$F * \delta(\pi, \sigma) = \sum_{\rho \in P, \pi \leq \rho \leq \sigma} F(\pi, \rho) \delta(\rho, \sigma) = F(\pi, \sigma), \forall F.$$

It follows that δ is the unit of set of all functions on $P^{(2)}$ with respect to convolution $*$. The inverse function of the function $\zeta : P^{(2)} \rightarrow \mathbb{C}$, $\zeta(\pi, \sigma) = 1, \forall (\pi, \sigma) \in P^{(2)}$, with respect to the convolution $*$ is called the Mobius function μ_P of P .

Free cumulants. Let $\pi, \sigma \in NC(n)$. We say $\pi \leq \sigma$ if each block (a subset of $[n]$) of π is completely contained in one of the blocks of σ . $NC(n)$ is a poset by this partial order. The Mobius function of $NC(n)$ is denoted by μ_n . The unital linear functional $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ produces a sequence of multilinear functionals

$$\varphi_n : \mathcal{A}^n \rightarrow \mathbb{C}, \varphi_n(a_1, a_2, \dots, a_n) = \varphi(a_1 a_2 \cdots a_n), n = 1, 2, \dots.$$

Let $V = \{i_1, i_2, \dots, i_s\} \subseteq [n]$ such that $i_1 < i_2 < \dots < i_s$. We define $\varphi_V(a_1, a_2, \dots, a_n) = \varphi(a_{i_1} a_{i_2} \cdots a_{i_s})$. More generally, for a partition $\pi = \{V_1, V_2, \dots, V_r\} \in NC(n)$, we define

$$\varphi_\pi(a_1, a_2, \dots, a_n) = \prod_{i=1}^r \varphi_{V_i}(a_1, a_2, \dots, a_n).$$

The n -th free cumulant of (\mathcal{A}, φ) is the multilinear functional $\kappa_n : \mathcal{A}^n \rightarrow \mathbb{C}$ defined by

$$\kappa_n(a_1, a_2, \dots, a_n) = \sum_{\pi \in NC(n)} \varphi_\pi(a_1, a_2, \dots, a_n) \mu_n(\pi, 1_n),$$

where $1_n = [n]$ is the single-block partition of $[n]$. The convergence in distribution of a sequence of families of random variables can be characterized in their free cumulants. A sequence $\{\{a_{i,n} : i \in I\} : n = 1, 2, \dots\}$ converge in distribution to $\{b_i : i \in I\}$ if and only if for all $i_1, i_2, \dots, i_m \in I$, $m \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} \kappa_m(a_{i_1,n}, a_{i_2,n}, \dots, a_{i_m,n}) = \kappa_m(b_{i_1}, b_{i_2}, \dots, b_{i_m}).$$

Free cumulants $\kappa_n : \mathcal{A}^n \rightarrow \mathbb{C}$ and free independence have a very beautiful relation.

Theorem 1.1 (Theorem 11.20 in [NS]). *A family $\{a_i : i \in I\}$ of elements in (\mathcal{A}, φ) is freely independent if and only if for all $n \geq 2$ and all $i(1), i(2), \dots, i(n) \in I$,*

$$\kappa_n(a_{i(1)}, a_{i(2)}, \dots, a_{i(n)}) = 0$$

whenever there exist $1 \leq l, k \leq n$ with $i(l) \neq i(k)$.

Semicircle families. Let (\mathcal{A}, φ) be a $*$ -probability space. A self-adjoint element $a \in \mathcal{A}$ is a *semicircle element* (or has a *semicircle distribution*) if

$$\varphi(a^n) = \frac{2}{\pi r^2} \int_{-r}^r t^n \sqrt{r^2 - t^2} dt, n = 1, 2, \dots,$$

where r is called the radius of the distribution of a . When $r = 2$, $\varphi(a^2) = 1$, we call a a *standard semicircle element* (or has a *standard semicircle distribution*). A semicircle element can be characterized by its moments $\varphi(a^{2k}) = (r^2/4)^k C_k$, where C_k is the k -th Catalan number, and $\varphi(a^{2k+1}) = 0$, $k = 0, 1, 2, \dots$, or by its free cumulants $\kappa_n(a) = \delta_{n,2} \frac{r^2}{4}$ ((11.13) in [NS]). Given an index set I and a positive definite matrix $(c_{ij})_{i,j \in I}$, a family $(s_i)_{i \in I}$ of self-adjoint random variables in a $*$ -probability space (\mathcal{A}, φ) is called a *semicircular family of covariance $(c_{ij})_{i,j \in I}$* if $\kappa_n(s_{i(1)}, \dots, s_{i(n)}) = \delta_{n,2} c_{i(1), i(2)}$, for $i(1), \dots, i(n) \in I$. (See 8.15 in [NS].)

2. MULTIDIMENSIONAL FREE POISSON DISTRIBUTIONS

By the discussion in Page 203 and Exercise 12.22 of [NS], a classical Poisson distribution is the limit in distribution of a sequence of convolutions of Bernoulli distributions. In the point of view of random variables, we can restate it as follows. Let $\lambda > 0, \alpha \in \mathbb{R}$. For each $N \in \mathbb{N}, N > \lambda$, let $\{b_{i,N} : i = 1, 2, \dots, N\}$ be a sequence of i.i.d. Bernoulli random variables such that

$$Pr(b_{i,N} = 0) = 1 - \frac{\lambda}{N}, Pr(b_{i,N} = \alpha) = \frac{\lambda}{N}.$$

Then the binomial random variable $S_N = \sum_{i=1}^N b_{i,N}$ has a binomial distribution

$$Pr(S_N = k\alpha) = C_N^k \left(\frac{\lambda}{N}\right)^k \left(1 - \frac{\lambda}{N}\right)^{N-k},$$

$k = 0, 1, 2, \dots, N$, where C_N^k is the combination number (or the binomial coefficient). Let $N \rightarrow \infty$, by elementary calculus, we can get

$$\lim_{N \rightarrow \infty} Pr(S_N = k\alpha) = \frac{\lambda^k}{k!} e^{-\lambda} = Pr(P = k\alpha),$$

where P has a Poisson distribution $Pr(P = k\alpha) = \frac{\lambda^k}{k!} e^{-\lambda}$, $k = 0, 1, 2, \dots$.

In the non-commutative case, the free Poisson limit theorem (Proposition 12.11 in [NS]) states that a free Poisson distribution is the limit in distribution of a sequence of free convolutions of Bernoulli distributions. Let's restate it in the language of random variables.

Let (\mathcal{A}, φ) be a non-commutative probability space. A Bernoulli random variable $a \in \mathcal{A}$ is a linear combination $a = \alpha p + \beta(1 - p)$, where $\alpha, \beta \in \mathbb{R}$, and $p \in \mathcal{A}$ is an idempotent (i. e., $p^2 = p$) with $0 \leq \varphi(p) \leq 1$. The classical interpretation of a Bernoulli random variable is that a is a random variable with two "values": α and β , and $Pr(a = \alpha) = \varphi(p)$, $Pr(a = \beta) = 1 - \varphi(p)$. In the free Poisson limit theorem, $\beta = 0$, $\varphi(p) = \frac{\lambda}{N}$, $N > \lambda$. We can restate the free Poisson limit theorem as follows. Let $\lambda > 0, \alpha \in \mathbb{R}$. For $N \in \mathbb{N}, N > \lambda$, let $\{\alpha p_{1,N}, \alpha p_{2,N}, \dots, \alpha p_{N,N}\}$ be a free family of Bernoulli random variables such that $\varphi(p_{i,N}) = \frac{\lambda}{N}$, $i = 1, 2, \dots, N$. Let $S_N = \sum_{i=1}^N \alpha p_{i,N}$. Then

$$\lim_{N \rightarrow \infty} \kappa_m(S_N) = \lambda \alpha^m, m = 1, 2, \dots.$$

Hence, we can restate the definition of free Poisson distributions as follows.

Definition 2.1 (Proposition 12.11, Definition 12.12 [NS]). *Let $\lambda \geq 0, \alpha \in \mathbb{R}$, and (\mathcal{A}, φ) be a non-commutative probability space. A random variable $a \in \mathcal{A}$ has a free Poisson distribution if the free cumulants of a are $\kappa_n(a) = \lambda \alpha^n, \forall n \in \mathbb{N}$.*

In this section, we shall generalize the results on free Poisson distributions in Lecture 12 of [NS] to the multidimensional case.

By the proof of Theorem 13.1 in [NS], we can modify the theorem slightly as follows.

Proposition 2.2 (Theorem 13.1 and Lemma 13.2 in [NS]). *Let $\{n_k\}$ be a sequence of natural numbers such that $\lim_{k \rightarrow \infty} n_k = \infty$, and, for each natural number k , $(\mathcal{A}_k, \varphi_k)$ be a non-commutative probability space. Let I be an index set. Consider a triangular array of random variables, i. e., for each $i \in I$, $0 \leq r \leq n_k$, we have a random variable $a_{n_k, r}^{(i)} \in \mathcal{A}_k$. Assume that, for each k , the sets $\{a_{(n_k, 1)}^{(i)}\}_{i \in I}, \{a_{(n_k, 2)}^{(i)}\}_{i \in I}, \dots, \{a_{(n_k, n_k)}^{(i)}\}_{i \in I}$ are free and identically distributed. Then the following statements are equivalent.*

- (1) *There is a family of random variables $(b_i)_{i \in I}$ in some non-commutative probability space (\mathcal{A}, φ) such that $(a_{n_k, 1}^{(i)} + a_{n_k, 2}^{(i)} + \dots + a_{n_k, n_k}^{(i)})_{i \in I}$ converges in distribution to $(b_i)_{i \in I}$, as $k \rightarrow \infty$.*
- (2) *For all $n \geq 1$, and all $i(1), i(2), \dots, i(n) \in I$, the limits $\lim_{k \rightarrow \infty} n_k \varphi_k(a_{n_k, r}^{(i(1))} \dots a_{n_k, r}^{(i(n))})$ exist, $1 \leq r \leq n_k$.*

- (3) For all $n \geq 1$, and all $i(1), i(2), \dots, i(n) \in I$, the limits $\lim_{k \rightarrow \infty} n_k \kappa_n^k(a_{n_k, r}^{(i(1))} \dots a_{n_k, r}^{(i(n))})$ exist, $1 \leq r \leq n_k$, where κ_n^k is the n -th free cumulant of \mathcal{A}_k .

Furthermore, if one of these conditions is satisfied, then the limits in (2) are equal to the corresponding limits in (3), and the joint distribution of the limit family $(b_i)_{i \in I}$ is determined by, for $n \geq 1, i(1), i(2), \dots, i(n) \in I$,

$$\kappa_n(b_{i(1)} b_{i(2)} \dots b_{i(n)}) = \lim_{k \rightarrow \infty} n_k \varphi_k(a_{n_k, r}^{(i(1))} a_{n_k, r}^{(i(2))} \dots a_{n_k, r}^{(i(n))}).$$

We will use the following elementary results in sequel.

Lemma 2.3. Let $\{a_{i,j} : i, j = 1, 2, \dots\}$ be a bi-index sequence of complex numbers. If $\sup\{|a_{i,j}| : i = 1, 2, \dots\} = M_j < \infty, \forall j$, then there exists a sequence $(n_k)_{k \in \mathbb{N}}$ of natural numbers such that $\lim_{k \rightarrow \infty} n_k = \infty$, and $\lim_{k \rightarrow \infty} a_{n_k, j}$ exists, $\forall j \in \mathbb{N}$.

Proof. Since $\{|a_{i,1}| : i = 1, 2, \dots\}$ is bounded, there is a sequence $\{i(k, 1) : k = 1, 2, \dots\}$ of natural numbers such that $\lim_{k \rightarrow \infty} a_{i(k, 1), 1} = a_1$, for some number a_1 . Consider the sequence $\{a_{i(k, 1), 2} : k = 1, 2, \dots\}$. Since the sequence is bounded, there is a subsequence $\{i(k, 2) : k = 1, 2, \dots\}$ of $\{i(k, 1) : k = 1, 2, \dots\}$ such that $\lim_{k \rightarrow \infty} a_{i(k, 2), 2} = a_2$. But we also have $\lim_{k \rightarrow \infty} a_{i(k, 2), 1} = a_1$. Continuing the process, we can obtain a bi-index sequence $\{i(k, l) : k, l = 1, 2, \dots\}$ of natural numbers such that $\lim_{k \rightarrow \infty} a_{i(k, l), j} = a_j$, for $j \leq l$. Let $n_k = i(k, k)$, for $k = 1, 2, \dots$. Then, for a j , and an $\epsilon > 0$, there exists a natural number $K > j$ such that $|a_{i(k, j), j} - a_j| < \epsilon, \forall k > K$. Note that $\{i(n, k) : n = 1, 2, \dots\}$ is a subsequence of $\{i(n, j) : n = 1, 2, \dots\}$. Thus, $i(k, k) \geq i(k, j)$, and $|a_{i(k, k)} - a_j| < \epsilon, \forall k > K$. It means that $\lim_{k \rightarrow \infty} a_{n_k, j} = a_j, \forall j$. \square

We need to review some basic concepts on ultrafilters of \mathbb{N} (See [GH] and Appendix A of [SS] for details). A filter \mathcal{F} is a non-empty set of subsets of \mathbb{N} satisfying

- (1) if $X \in \mathcal{F}$ and $X \subseteq Y \subseteq \mathbb{N}$, then $Y \in \mathcal{F}$;
- (2) if $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$;
- (3) \emptyset is not in \mathcal{F} .

A filter \mathcal{F} is an *ultrafilter* if for $\forall X \subseteq \mathbb{N}$, we have $X \in \mathcal{F}$ or $\mathbb{N} \setminus X \in \mathcal{F}$. An ultrafilter is called a *free ultrafilter* if \mathcal{F} does not contain any finite sets. A filter \mathcal{F} is *infinite* if V is an infinite subset, $\forall V \in \mathcal{F}$. Let $\beta\mathbb{N}$ be the Stone-Cech compactification of the set \mathbb{N} . Free ultrafilters can be identified as points in $\beta\mathbb{N} \setminus \mathbb{N}$. For a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, a sequence $a = (a_n)_n \in l^\infty(\mathbb{N})$, and a number $l \in \mathbb{C}$, we say that $\lim_{n \rightarrow \omega} a_n = l$ if for $\forall \epsilon > 0$, we have $\{n : |x_n - l| < \epsilon\} \in \omega$.

Lemma 2.4. Let $x = (x_n) \in l^\infty(\mathbb{N})$ and $l \in \mathbb{C}$. Then there is a subsequence $\{n_k : k = 1, 2, \dots\}$ of natural numbers such that $\lim_{k \rightarrow \infty} x_{n_k} = l$ if and only if there is a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ such that $\lim_{n \rightarrow \omega} x_n = l$.

Proof. If $\lim_{n \rightarrow \omega} x_n = l$, by the definition, for any $\epsilon > 0$, $\{n : |x_n - l| < \epsilon\} \in \omega$. Let $F_k = \{n : |x_n - l| < \frac{1}{k}\} \in \omega$. There exists $n_1 \in F_1$. Since $F_2 = \{n : |x_n - l| < \frac{1}{2}\} \in \omega$ is an infinite set, there is an $n_2 \in \{n > n_1 : n \in F_2\}$. Inductively, we get sequence $n_1 < n_2 < \dots < n_k < \dots$ such that $n_k \in F_k$, that is, $|x_{n_k} - l| < \frac{1}{k}$, for $k = 1, 2, \dots$. It follows that $\lim_{k \rightarrow \infty} x_{n_k} = l$.

Conversely, if $\lim_{k \rightarrow \infty} x_{n_k} = l$, then $\mathcal{F}_0 = \{\{n : |x_n - l| < \frac{1}{k}\} : k = 1, 2, \dots\}$ is a set of subsets of \mathbb{N} having the finite intersection property (i.e., the intersection of any finitely many subsets in \mathcal{F}_0 is not empty). Moreover, any such an intersection is an infinite set in \mathcal{F}_0 . Let

$$\mathcal{F} = \{X \subseteq \mathbb{N} : \exists k_0 \in \mathbb{N}, \{n : |x_n - l| < \frac{1}{k_0}\} \subseteq X\}.$$

It is obvious that \mathcal{F} is an infinite filter. The set \mathfrak{F} of all infinite filters \mathcal{S} of \mathbb{N} containing \mathcal{F} (i. e., $\mathcal{F} \subseteq \mathcal{S}$) is a poset (partial ordered set) with respect to the inclusion of sets. By Zorn's lemma, there is a maximal filter ω in \mathfrak{F} , which is a free ultrafilter (see [AK] for the details). For $\forall \epsilon > 0$, there exists a $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0} < \epsilon$. Therefore, $\{n : |x_n - l| < \epsilon\} \supseteq \{n : |x_n - l| < \frac{1}{k_0}\}$. It follows that $\{n : |x_n - l| < \epsilon\} \in \mathcal{F} \subseteq \omega$, that is, $\lim_{n \rightarrow \omega} x_n = l$. \square

Theorem 2.5. Let $\{\alpha_i : i = 1, 2, \dots\}$ be a sequence of real numbers, $\{\lambda_i > 0\}_{i \in \mathbb{N}}$ with $\lambda = \sup\{\lambda_i : i \geq 1\} < \infty$, and for each $N \in \mathbb{N}, N \geq \lambda$, there be N freely independent and identically distributed sequences

$$\{p_{1,N}^{(i)}\}_{i \in \mathbb{N}}, \{p_{2,N}^{(i)}\}_{i \in \mathbb{N}}, \dots, \{p_{N,N}^{(i)}\}_{i \in \mathbb{N}}$$

of projections on a C^* -probability space $(\mathcal{A}_N, \varphi_N)$. Moreover, $\varphi_N(p_{r,N}^{(i)}) = \frac{\lambda_i}{N}, i = 1, 2, \dots, r = 1, 2, \dots, N$. Define a triangular family of sequences of random variables $\{a_{j,N}^{(i)} = \alpha_i p_{j,N}^{(i)} : i = 1, 2, \dots\}$, for $j = 1, 2, \dots, N, N \in \mathbb{N}, N \geq \lambda$. Then there exists a family of random variables $(b_i)_{i \in \mathbb{N}}$ in a C^* -probability space (\mathcal{A}, φ) and a sequence $\{n_k : k = 1, 2, \dots\}$ of natural numbers such that $\lim_{k \rightarrow \infty} n_k = \infty$ and $(a_{1,n_k}^{(i)} + a_{2,n_k}^{(i)} + \dots + a_{n_k,n_k}^{(i)})_{i \in \mathbb{N}}$ converges to $(b_i)_{i \in \mathbb{N}}$ in distribution, as $k \rightarrow \infty$. Moreover, for $i_1, i_2, \dots, i_n, n \in \mathbb{N}$,

$$\kappa_n(b_{i_1}, b_{i_2}, \dots, b_{i_n}) = \alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_n} \lim_{k \rightarrow \infty} n_k \varphi(p_{1,n_k}^{(i_1)} p_{1,n_k}^{(i_2)} \dots p_{1,n_k}^{(i_n)}).$$

Proof. For $N, i(1), i(2), \dots, i(n), n \in \mathbb{N}$, let

$$f(N, i(1), i(2), \dots, i(n)) = N \varphi_N(a_{r,N}^{(i(1))} a_{r,N}^{(i(2))} \dots a_{r,N}^{(i(n))}), 1 \leq r \leq N,$$

and $M(i(1), i(2), \dots, i(n)) = |\alpha_{i(1)} \alpha_{i(2)} \dots \alpha_{i(n)}| \lambda_{i(1)}^{1/2} \lambda_{i(n)}^{1/2}$. Then

$$\begin{aligned} |f(N, i(1), i(2), \dots, i(n))| &= N |\alpha_{i(1)} \alpha_{i(2)} \dots \alpha_{i(n)} \varphi_N(p_{r,N}^{(i(1))} p_{r,N}^{(i(2))} \dots p_{r,N}^{(i(n))})| \\ &\leq N |\alpha_{i(1)} \alpha_{i(2)} \dots \alpha_{i(n)}| \varphi_N(p_{r,N}^{(i(1))})^{1/2} \varphi_N(p_{r,N}^{(i(n))})^{1/2} \dots \varphi_N(p_{r,N}^{(i(3))})^{1/2} \varphi_N(p_{r,N}^{(i(2))})^{1/2} \varphi_N(p_{r,N}^{(i(1))})^{1/2} \\ &\leq N |\alpha_{i(1)} \alpha_{i(2)} \dots \alpha_{i(n)}| \frac{\lambda_{i(1)}^{1/2}}{N^{1/2}} \varphi_N(p_{r,N}^{(i(n))})^{1/2} \dots \varphi_N(p_{r,N}^{(i(4))})^{1/2} \varphi_N(p_{r,N}^{(i(3))})^{1/2} \varphi_N(p_{r,N}^{(i(2))})^{1/2} \varphi_N(p_{r,N}^{(i(1))})^{1/2} \\ &\leq \dots \\ &\leq N |\alpha_{i(1)} \alpha_{i(2)} \dots \alpha_{i(n)}| \frac{\lambda_{i(1)}^{1/2}}{N^{1/2}} \varphi_N(p_{r,N}^{(i(n))})^{1/2} \\ &= N |\alpha_{i(1)} \alpha_{i(2)} \dots \alpha_{i(n)}| \frac{\lambda_{i(1)}^{1/2}}{N^{1/2}} \frac{\lambda_{i(n)}^{1/2}}{N^{1/2}} \\ &= M(i(1), i(2), \dots, i(n)). \end{aligned}$$

Let

$$S_m = \{(i(1), i(2), \dots, i(n)) : i(1) + i(2) + \dots + i(n) = m, i(1), i(2), \dots, i(n) \in \mathbb{N}\},$$

for $m \in \mathbb{N}$. Then for each $m \in \mathbb{N}$, S_m is a finite set with $|S_m| = k_m$, and $\{S_m : m \in \mathbb{N}\}$ is a partition of the set $\{(i(1), i(2), \dots, i(n)) : i(1), i(2), \dots, i(n), n \in \mathbb{N}\}$. Define a bijective map

$$\gamma : S_1 \rightarrow \{1\}, \gamma : S_m \rightarrow \{(\sum_{l=1}^{m-1} k_l) + 1, (\sum_{l=1}^{m-1} k_l) + 2, \dots, \sum_{l=1}^m k_l\}, m \geq 2.$$

For instance, $\gamma((1, 1)) = 2, \gamma(2) = 3, \gamma(S_2) = \{2, 3\}$. It implies that

$$\begin{aligned} &\gamma(\{(i(1), i(2), \dots, i(n)) : i(1), i(2), \dots, i(n), n \in \mathbb{N}\}) \\ &= \gamma(S_1) \cup \gamma(S_2) \cup \dots \cup \gamma(S_m) \cup \dots \\ &= \{j : j = 1, 2, 3, \dots\}. \end{aligned}$$

Thus, $\{f(N, i(1), i(2), \dots, i(n)) : N, i(1), i(2), \dots, i(n), n \in \mathbb{N}\} = \{f(N, \gamma^{-1}(j)) : N, j \in \mathbb{N}\}$ is a bi-index sequence. By Lemma 2.3, there is a sequence $(n_k)_{k \in \mathbb{N}}$ of natural numbers such that $\lim_{k \rightarrow \infty} n_k = \infty$, and $f(n_k, i(1), i(2), \dots, i(n))$ converges as $k \rightarrow \infty$, for every tuple $(i(1), i(2), \dots, i(n))$. By Theorem 2.2, there is a family of random variables $(b_i)_{i \in \mathbb{N}}$ in a non-commutative probability

space (\mathcal{A}, φ) such that $((a_{1,n_k}^{(i)} + a_{2,n_k}^{(i)} + \cdots + a_{n_k,n_k}^{(i)})_{i \in \mathbb{N}})$ converges to $(b_i)_{i \in \mathbb{N}}$ in distribution. Moreover, for $i_1, i_2, \dots, i_n, n \in \mathbb{N}$,

$$\kappa_m(b_{i_1}, b_{i_2}, \dots, b_{i_n}) = \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_n} \lim_{k \rightarrow \infty} n_k \varphi(p_{1,n_k}^{(i_1)} p_{1,n_k}^{(i_2)} \cdots p_{1,n_k}^{(i_n)}).$$

Now we shall construct a C^* -probability space (\mathcal{A}, φ) and $b_i \in \mathcal{A}, i = 1, 2, \dots$, such that $(b_i)_i$ has the limit distribution. Let $s_N^{(i)} = \sum_{j=1}^N a_{j,N}^{(i)} \in \mathcal{A}_N$, for $N > \lambda, i = 1, 2, \dots$. We have proved that there is a subsequence $\{n_k : k = 1, 2, \dots\}$ such that $\lim_{k \rightarrow \infty} \varphi_{n_k}(s_{n_k}^{(i(1))} s_{n_k}^{(i(2))} \cdots s_{n_k}^{(i(n))})$ converges for $i(1), i(2), \dots, i(n) \in \mathbb{N}$. By Lemma 2.4, there is a $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ such that

$$\lim_{N \rightarrow \omega} \varphi_N(s_N^{(i(1))} s_N^{(i(2))} \cdots s_N^{(i(n))}) = \lim_{k \rightarrow \infty} \varphi_{n_k}(s_{n_k}^{(i(1))} s_{n_k}^{(i(2))} \cdots s_{n_k}^{(i(n))}), \forall i(1), i(2), \dots, i(n) \in \mathbb{N}. \quad (1.1)$$

Let $\mathcal{M} = \bigoplus_{N \in \mathbb{N}} \mathcal{A}_N$ be the l^∞ -product of \mathcal{A}_N 's. Let $I_\omega = \{x = (x_N) \in \mathcal{M} : \lim_{N \rightarrow \omega} \|x_N\| = 0\}$. Then $\mathcal{A} = \prod_{N \in \mathbb{N}} \mathcal{A}_N / I_\omega$ is a unital C^* -algebra, which is called the ultra-product C^* -algebra of \mathcal{A}_N 's (see Section 2 of [GH] for details). Since $(\varphi_N(x_N))_N \in l^\infty(\mathbb{N})$, $\lim_{N \rightarrow \omega} \varphi_N(x_N)$ exists. We define $\varphi_\omega : \mathcal{A} \rightarrow \mathbb{C}$ by

$$\varphi_\omega(x) = \lim_{N \rightarrow \omega} \varphi_N(x_N), \forall x = (x_N)_N \in \mathcal{A}.$$

We show that φ_ω is a state on \mathcal{A} . It is obvious that $\varphi_\omega(1) = \lim_{N \rightarrow \omega} \varphi_N(1_N) = 1$ and

$$\varphi_\omega(x^*x) = \lim_{N \rightarrow \omega} \varphi_N(x_N^*x_N) \geq 0, \forall x = (x_N)_N \in \mathcal{A}.$$

Therefore, $(\mathcal{A}, \varphi_\omega)$ is a C^* -probability space. Let $b_i = (s_N^{(i)})_N \in \mathcal{A}$, for $i \in \mathbb{N}$. Then

$$\varphi_\omega(b_{i(1)} b_{i(2)} \cdots b_{i(n)}) = \lim_{N \rightarrow \omega} \varphi_N(s_N^{(i(1))} s_N^{(i(2))} \cdots s_N^{(i(n))}), \forall i(1), i(2), \dots, i(n) \in \mathbb{N}. \quad (1.2)$$

By (1.1) and (1.2), the sequence $\{b_i : i \geq 1\}$ has the limit distribution. \square

We call the limit distribution in Theorem 2.5 a multidimensional free Poisson distributions. Precisely, we have the following definition.

Definition 2.6. Let $\{\alpha_i : i = 1, 2, \dots\}$ be a sequence of real numbers, $\{\lambda_i > 0 : i = 1, 2, \dots\}$ with $\lambda = \sup\{\lambda_i : i \geq 1\} < \infty$. A sequence $\{b_i : i = 1, 2, \dots\}$ of random variables in a non-commutative probability space (\mathcal{B}, ϕ) has a multidimensional free Poisson distribution, if there is a $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, and, for each $N \in \mathbb{N}, N \geq \lambda$, there is a family $\{p_N^{(i)} : i \in \mathbb{N}\}$ of projections in a C^* -probability space $(\mathcal{A}_N, \varphi_N)$ such that $\varphi_N(p_N^{(i)}) = \frac{\lambda_i}{N}$, and

$$\kappa_n(b_{i(1)}, b_{i(2)}, \dots, b_{i(n)}) = \alpha_{i(1)} \alpha_{i(2)} \cdots \alpha_{i(n)} \lim_{N \rightarrow \omega} N \varphi_N(p_N^{(i(1))} p_N^{(i(2))} \cdots p_N^{(i(n))}),$$

for all $(i(1), i(2), \dots, i(n)) \in \mathbb{N}^n$.

Remark 2.7. (1) By Theorems 2.2 and 2.5, For each $i \in \mathbb{N}$,

$$\kappa_n(b_i) = \lim_{k \rightarrow \infty} n_k \varphi_{n_k}((a_{r,n_k}^{(i)})^n) = \alpha_i^n \lambda_i, n = 1, 2, \dots.$$

Hence, b_i has a free Poisson distribution, for each $i \in \mathbb{N}$.

(2) If $\sum_{i=1}^\infty \lambda_i < \infty$, $\{p_{r,N}^{(i)}\}_{i \in \mathbb{N}}$ is an orthogonal sequence of projections in a W^* -probability space, then, for $N > \sum_{i=1}^\infty \lambda_i, r = 1, 2, \dots, N$,

$$\kappa_n(b_{i(1)} b_{i(2)} \cdots b_{i(n)}) = \lim_{k \rightarrow \infty} n_k \varphi_{n_k}(\alpha_{i(1)} \cdots \alpha_{i(n)} p_{r,n_k}^{(i(1))} \cdots p_{r,n_k}^{(i(n))}) = 0,$$

whenever there are $i(j) \neq i(l), 0 \leq j, l \leq n$. This means that $\{b_i : i \in \mathbb{N}\}$ is a free family of free Poisson random variables. A similar procedure of constructing a free family from an orthogonal one can be found in Example 12.19 in [NS].

(3) Combining above (1) and (2), we get a conclusion. If $\{b_i : i = 1, 2, \dots\}$ is a free sequence of free Poisson random variables with $\kappa_n(a_i) = \alpha_i^n \lambda_i$, for $n, i = 1, 2, \dots$, and $\sum_{i=1}^\infty \lambda_i = \lambda < \infty$, then $\{b_i : i = 1, 2, \dots\}$ has a multidimensional free Poisson distribution.

Now we give a multidimensional limit theorem of compound free Poisson distributions.

Theorem 2.8. Let $\{a_i : i = 1, 2, \dots\}$ be a sequence of self-adjoint operators in a C^* -probability space (\mathcal{A}, φ) and $\{\lambda_i > 0 : i = 1, 2, \dots\}$ with $\sup\{\lambda_i : i = 1, 2, \dots\} = \lambda < \infty$. For each $N \in \mathbb{N}$ and $N \geq \lambda$, Let (\mathcal{C}_N, ψ) be a C^* -probability space having projections $p_N^{(i)} \in \mathcal{C}_N$ with $\psi(p_N^{(i)}) = \frac{\lambda_i}{N}$, for $i \in \mathbb{N}$, and $\mathcal{A}_N = \mathcal{A} \otimes \mathcal{C}_N$ be the spacial tensor product of \mathcal{A} and \mathcal{C}_N with a state $\varphi_N = \varphi \otimes \psi$. Let $a_N^{(i)} = a_i \otimes p_N^{(i)}$, and $\{a_{N,j}^{(i)} = a_i \otimes p_{N,j}^{(i)} : i = 1, 2, \dots, \} : j = 1, 2, \dots, N\}$ be a freely independent family of N identically distributed sequences of self-adjoint operators in \mathcal{A}_N such that the distribution of $\{a_{N,j}^{(i)} : i = 1, 2, \dots\}$ is same as that of $\{a_N^{(i)} : i = 1, 2, \dots\}$, for $j = 1, 2, \dots, N$. Let, moreover, $s_N^{(i)} = \sum_{j=1}^N a_{N,j}^{(i)}$. Then there is a sequence $\{b_i : i = 1, 2, \dots\}$ of compound free Poisson random variables in a C^* -probability space (\mathcal{B}, ϕ) and an $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ such that $\{s_N^{(i)}\}$ converges in distribution to $\{b_i; i = 1, 2, \dots\}$, as $N \rightarrow \omega$, and

$$\kappa_n(b_{i(1)}, b_{i(2)}, \dots, b_{i(n)}) = \varphi(a_{i(1)} a_{i(2)} \cdots a_{i(n)}) \lim_{N \rightarrow \omega} N \varphi_N(p_N^{(i(1))} p_N^{(i(2))} \cdots p_N^{(i(n))}),$$

for all $(i(1), i(2), \dots, i(n)) \in \mathbb{N}^n$.

Remark 2.9. The Proof of the above theorem is same as that of Theorem 2.5. The only change is to let $M(i(1), i(2), \dots, i(n)) = |\varphi(a_{i(1)} a_{i(2)} \cdots a_{i(n)})| \lambda_{i(1)}^{1/2} \lambda_{i(2)}^{1/2} \cdots \lambda_{i(n)}^{1/2}$ in the proof of Theorem 2.5, where $(i(1), i(2), \dots, i(n)) \in \mathbb{N}^n$, $n \in \mathbb{N}$.

Definition 2.10. Let $\{a_i : i = 1, 2, \dots\}$ be a sequence of self-adjoint operators in a C^* -probability space (\mathcal{A}, φ) and $\{\lambda_i > 0 : i = 1, 2, \dots\}$ with $\lambda = \sup\{\lambda_i : i \geq 1\} < \infty$. A sequence $\{b_i : i = 1, 2, \dots\}$ of random variables in a non-commutative probability space (\mathcal{B}, ϕ) has a multidimensional compound free Poisson distribution, if there is a $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, and for each $N \in \mathbb{N}$, $N \geq \lambda$, there is a family $\{p_N^{(i)} : i \in \mathbb{N}\}$ of projections in a C^* -probability space $(\mathcal{A}_N, \varphi_N)$ such that $\varphi_N(p_N^{(i)}) = \frac{\lambda_i}{N}$, and

$$\kappa_n(b_{i(1)}, b_{i(2)}, \dots, b_{i(n)}) = \varphi(a_{i(1)} a_{i(2)} \cdots a_{i(n)}) \lim_{N \rightarrow \omega} N \varphi_N(p_N^{(i(1))} p_N^{(i(2))} \cdots p_N^{(i(n))}),$$

for all $(i(1), i(2), \dots, i(n)) \in \mathbb{N}^n$.

Example 2.11. (1) For a positive number λ , and a sequence $\{a_1, a_2, \dots\}$ of self-adjoint operators in a C^* -probability space (\mathcal{A}, φ) . Let $p_N^{(i)} = p_N$, for $i = 1, 2, \dots$, with $\psi(p_N) = \frac{\lambda}{N}$ in Theorem 2.8. Then the limit random variable family $\{b_1, b_2, \dots, b_m\}$ has the n -th free cumulant

$$\kappa_n(b_{i(1)}, b_{i(2)}, \dots, b_{i(n)}) = \lambda \varphi(a_{i(1)} a_{i(2)} \cdots a_{i(n)}),$$

for $i(1), i(2), \dots, i(n) \in \{1, 2, \dots\}$, and $n \in \mathbb{N}$. It follows that the sequence $\{b_i : i = 1, 2, \dots\}$ with free cumulants $\kappa_n(b_{i(1)}, \dots, b_{i(n)}) = \lambda \varphi(a_{i(1)} \cdots a_{i(n)})$, for $i(1), i(2), \dots, i(n) \in \{1, 2, \dots\}$, and $n \in \mathbb{N}$, has a multidimensional compound free Poisson distribution. When $\{a_i : i = 1, 2, \dots\}$ is a finite sequence (i.e., $a_n = 0$, for all $n > m$, for some $m \in \mathbb{N}$), this is the limit theorem 4.4.3 in [RS] when $B = \mathbb{C}$, and the limit distribution is the (multidimensional) compound B -Poisson distribution defined in 4.4.1 in [RS] when $B = \mathbb{C}$.

(2) When $\lambda = 1$ in the above example, we get

$$\kappa_n(b_{i(1)}, b_{i(2)}, \dots, b_{i(n)}) = \varphi(a_{i(1)} a_{i(2)} \cdots a_{i(n)}),$$

for $i(1), i(2), \dots, i(n) \in \{1, 2, \dots, m\}$, and $n \in \mathbb{N}$, which is the n -th free cumulant of $\{sa_{i(1)}s, sa_{i(2)}s, \dots, sa_{i(n)}s\}$, where s is the semicircle random variable with $\varphi(s^2) = 1$, and is free from $\{a_n : n \in \mathbb{N}\}$ (see Example 12.19 in [NS]). Therefore, $\{sa_n s : n = 1, 2, \dots, m\}$ constructed in Example 12.19 in [NS] has a multidimensional compound free Poisson distribution.

(3) A natural question is to consider the distribution of $\{s_i a_i s_i : i = 1, 2, \dots\}$, where $\{s_i : i = 1, 2, \dots\}$ is a semicircle family, and $\{s_i : i = 1, 2, \dots\}$ is free from $\{a_i : i = 1, 2, \dots\}$. Let

$b_i = s_i a_i s_i$, for $i \in \mathbb{N}$. For $i(1), i(2), \dots, i(n) \in \mathbb{N}$, by the proof of Proposition 12.18 in [NS], for $n > 1$, we have

$$\begin{aligned}
& \kappa_n(b_{i(1)}, b_{i(2)}, \dots, b_{i(n)}) \\
&= \sum_{\pi \in NC(3n), \pi \vee \sigma = 1_{3n}} \kappa_\pi(s_{i(1)}, a_{i(1)}, s_{i(1)}, s_{i(2)}, a_{i(2)}, s_{i(2)}, \dots, s_{i(n)}, a_{i(n)}, s_{i(n)}) \\
&= \sum_{\pi \in NC(n)} \left(\prod_{j=1}^{n-1} \varphi(s_{i(j)} s_{i(j+1)}) \right) \varphi(s_{i(n)} s_{i(1)}) \kappa_\pi(a_{i(1)}, a_{i(2)}, \dots, a_{i(n)}) \\
&= \left(\prod_{j=1}^{n-1} \varphi(s_{i(j)} s_{i(j+1)}) \right) \varphi(s_{i(n)} s_{i(1)}) \varphi(a_{i(1)} a_{i(2)} \cdots a_{i(n)}),
\end{aligned}$$

where $\sigma = \{\{1, 2, 3\}, \{4, 5, 6\}, \dots, \{3n-2, 3n-1, 3n\}\}$. If $\{s_i : i = 1, 2, \dots\}$ is a free family of semicircle random variables, then $\kappa_n(b_{i(1)} b_{i(2)} \cdots b_{i(n)}) = 0$, if there are two j_1 and j_2 such that $i(j_1) \neq i(j_2)$. This shows that $\{b_i : i \in \mathbb{N}\}$ is a free family of compound free Poisson random variables.

- (4) Now consider a finite family $\{s_i a_i s_i : i = 1, 2, \dots, m\}$ of random variables, where $\{s_i : i = 1, 2, \dots, m\}$ is a free family of standard semicircle random variables, and the family is free from $\{a_i : i = 1, 2, \dots, m\}$. By above (3), $\{b_i = s_i a_i s_i : i = 1, 2, \dots, m\}$ is a free family of free compound free Poisson random variables. We want to show that $\{b_i, i = 1, 2, \dots, m\}$ has a multidimensional compound free Poisson distribution. In fact, for each natural number $N > m$, choose an orthogonal family $\{p_N^{(i)} : i = 1, 2, \dots, m\}$ of projections in a C^* -probability space (\mathcal{A}_N, φ) such that $\varphi_N(p_N^{(i)}) = 1/N$. It is obvious that

$$\kappa_n(b_{i(1)}, b_{i(2)}, \dots, b_{i(n)}) = \varphi(a_{i(1)} a_{i(2)} \cdots a_{i(n)}) \lim_{N \rightarrow \infty} N \varphi_N(p_N^{(i(1))} p_N^{(i(2))} \cdots p_N^{(i(n))}),$$

for all $(i(1), i(2), \dots, i(n)) \in \mathbb{N}^n$. Therefore, by Definition 2.10, $\{b_1, b_2, \dots, b_m\}$ has a multidimensional compound free Poisson distribution.

- (5) Let $\{b_i : i = 1, 2, \dots\}$ be a free family of compound free Poisson random variables with $\kappa_n(b_i) = \varphi(a_i^n) \lambda_i$, where $\{a_i : i = 1, 2, \dots\}$ be a sequence of self-adjoint operators in a C^* -probability space (\mathcal{A}, φ) , $\lambda_i > 0, i = 1, 2, \dots$, and $\sum_{i=1}^{\infty} \lambda_i = \lambda < \infty$. We show that $\{b_i : i = 1, 2, \dots\}$ has a multidimensional compound free Poisson distribution. For each $N > \lambda$, let $\{p_N^{(i)} : i = 1, 2, \dots\}$ be an orthogonal sequence of projections in a W^* -probability space (\mathcal{A}, φ) such that $\varphi_N(p_N^{(i)}) = \lambda_i/N$, for $i = 1, 2, \dots$. Then It is obvious that

$$\kappa_n(b_{i(1)}, b_{i(2)}, \dots, b_{i(n)}) = \varphi(a_{i(1)} a_{i(2)} \cdots a_{i(n)}) \lim_{N \rightarrow \infty} N \varphi_N(p_N^{(i(1))} p_N^{(i(2))} \cdots p_N^{(i(n))}),$$

for all $(i(1), i(2), \dots, i(n)) \in \mathbb{N}^n$. Therefore, by Definition 2.10, $\{b_1, b_2, \dots\}$ has a multidimensional compound free Poisson distribution.

3. MULTIDIMENSIONAL FREE INFINITELY DIVISIBLE DISTRIBUTIONS

In this section, we generalize the results on multidimensional free infinitely divisible distributions in [RS] and [BG] to a more general case.

Definition 3.1. A sequence $\{a_i : i \in \mathbb{N}\}$ of random variables in a non-commutative probability space (\mathcal{A}, φ) has a free infinitely divisible (joint) distribution if for every $k \in \mathbb{N}$, there are k freely independent identically distributed sequences $\{a_{j,k}^{(i)} : i \in \mathbb{N}\}, j = 1, 2, \dots, k$, such that the distribution of $\{\sum_{j=1}^k a_{j,k}^{(i)} : i \in \mathbb{N}\}$ is same as that of $\{a_i : i \in \mathbb{N}\}$. We call the distribution an infinite dimensional free infinitely divisible distribution.

A concept related to multidimensional free infinitely divisible distributions is multidimensional free Levy processes. A finite dimensional version of the following definition $\{a_t^{(i)} : t \geq 0, i = 1, 2, \dots, m\}$ is constructed in Section 4.7 in [RS].

Definition 3.2. A family $\{\{a_t^{(i)} : i = 1, 2, \dots\} t \geq 0\}$ of random variables in a non-commutative probability space (\mathcal{A}, φ) is a multidimensional free Levy processes if it satisfies the following conditions.

- (1) For $t > s \geq 0, v > u \geq 0$, $\{a_t^{(i)} - a_s^{(i)} : i = 1, 2, \dots\}$ and $\{a_v^{(i)} - a_u^{(i)} : i = 1, 2, \dots\}$ are free if $(s, t) \cap (u, v) = \emptyset$.
- (2) For $t > s \geq 0$, $\{a_t^{(i)} - a_s^{(i)} : i = 1, 2, \dots\}$ and $\{a_{t-s}^{(i)} : i = 1, 2, \dots\}$ have the same distribution.
- (3) $a_0^{(i)} = 0$, for $i = 1, 2, \dots$.
- (4) $\{a_t^{(i)} : i = 1, 2, \dots\}$ converges to 0 in distribution as $t \rightarrow 0$.

The following theorem is the main result of this section.

Theorem 3.3. Let $\{a_i : i \in \mathbb{N}\}$ be a sequence of self-adjoint operators in a C^* -probability space (\mathcal{A}, φ) . Then the first four of the following statements are equivalent. If, furthermore, the linear functional φ on \mathcal{A} is tracial (i. e., $\varphi(ab) = \varphi(ba)$, for all $a, b \in \mathcal{A}$), then the following five statements are equivalent.

- (1) $\{a_n : n \in \mathbb{N}\}$ has a distribution same as that of $\{a_1^{(i)} : i = 1, 2, \dots\}$ of a multidimensional free Levy process $\{\{a_t^{(i)} : i = 1, 2, \dots\} : t \geq 0\}$ of self-adjoint operators in a C^* -probability space (\mathcal{A}, φ) .
- (2) Let $\mathcal{P} = \mathbb{C}\langle X_i : i \in \mathbb{N} \rangle_0$ be the non-unital $*$ -algebra generated by non-commutative indeterminates X_1, X_2, \dots such that $X_i^* = X_i$ for $i \in \mathbb{N}$.
For $\underline{i} = (i(1), i(2), \dots, i(m)) \in \mathbb{N}^m, \underline{j} = (j(1), j(2), \dots, j(n)) \in \mathbb{N}^n$, let
 $\underline{j}_r = (j(n), j(n-1), \dots, j(1)), X_{\underline{i}} = X_{i(1)} X_{i(2)} \cdots X_{i(m)}, a_{\underline{i}} = (a_{i(1)}, \dots, a_{i(m)}).$
Define $\langle X_{\underline{i}}, X_{\underline{j}} \rangle = \kappa_{m+n}(a_{\underline{i}}, a_{\underline{j}_r})$. Then $\langle \cdot, \cdot \rangle$ is a non-negative sesquilinear form on \mathcal{P} .
- (3) For every $N \in \mathbb{N}$, there are N freely independent identically distributed sequences $\{a_{j,N}^{(i)} : i \in \mathbb{N}\}, j = 1, 2, \dots, N$, in a C^* -probability space $(\mathcal{A}_N, \varphi_N)$, and a subsequence $\{N_k : k \in \mathbb{N}\}$ of distinct natural numbers such that the distribution of $\{\sum_{j=1}^{N_k} a_{j,N_k}^{(i)} : i \in \mathbb{N}\}$ converges to the distribution of $\{a_i : i \in \mathbb{N}\}$, as $k \rightarrow \infty$.
- (4) $\{a_n : n = 1, 2, \dots\}$ has a free infinitely divisible distribution.
- (5) Define a linear functional $\kappa : \mathcal{P} \rightarrow \mathbb{C}$, $\kappa(P) = \sum \alpha_{i_1, \dots, i_n} \kappa_n(a_{i_1}, \dots, a_{i_n})$, for a polynomial $P = \sum \alpha_{i_1, \dots, i_n} X_{i_1} \cdots X_{i_n} \in \mathcal{P}$. Then κ is positive and tracial, and $\kappa(P^*) = \overline{\kappa(P)}$, for all $P \in \mathcal{P}$.

Proof. The implications from (4) to (3), and from (1) to (4) are obvious.

(3) \Rightarrow (2). By Theorem 2.2,

$$\kappa_{m+n}(a_{\underline{i}}, a_{\underline{j}_r}) = \lim_{k \rightarrow \infty} N_k \varphi_{N_k}(a_{i(1)} \cdots a_{i(m)} a_{j(n)} a_{i(n-1)} \cdots a_{j(1)}).$$

For a polynomial $P = \sum \alpha_{\underline{i}} X_{\underline{i}} \in \mathcal{P}$, we have

$$\begin{aligned} \langle P, P \rangle &= \sum \alpha_{\underline{i}} \overline{\alpha_{\underline{j}}} \langle X_{\underline{i}}, X_{\underline{j}} \rangle = \sum \alpha_{\underline{i}} \overline{\alpha_{\underline{j}}} \kappa_{m+n}(a_{\underline{i}}, a_{\underline{j}_r}) \\ &= \lim_{k \rightarrow \infty} \sum \alpha_{\underline{i}} \overline{\alpha_{\underline{j}}} N_k \varphi_{N_k}(a_{i(1)} \cdots a_{i(m)} a_{j(n)} \cdots a_{j(1)}) \\ &= \lim_{k \rightarrow \infty} N_k \varphi_{N_k}(P(a) P(a)^*) \geq 0, \end{aligned}$$

where $P(a) = \sum \alpha_{\underline{i}} a_{i(1)} \cdots a_{i(n)}$. Moreover, for $X_{\underline{i}} = X_{i(1)} X_{i(2)} \cdots X_{i(m)}$, $X_{\underline{j}} = X_{j(1)} X_{j(2)} \cdots X_{j(n)} \in \mathcal{P}$, we have

$$\begin{aligned} \langle X_{\underline{i}}, X_{\underline{j}} \rangle &= \kappa_{m+n}(a_{\underline{i}} a_{\underline{j}}) = \lim_{k \rightarrow \infty} N_k \varphi_{N_k}(a_{i(1)} \cdots a_{i(m)} a_{j(n)} \cdots a_{j(1)}) \\ &= \lim_{k \rightarrow \infty} \overline{N_k \varphi_{N_k}(a_{j(1)} \cdots a_{j(m)} a_{i(m)} \cdots a_{i(1)})} \\ &= \overline{\kappa_{m+n}(a_{\underline{j}}, a_{\underline{i}})} = \overline{\langle X_{\underline{j}}, X_{\underline{i}} \rangle}. \end{aligned}$$

It follows that $\langle \cdot, \cdot \rangle$ is a non-negative sesquilinear form on \mathcal{P} .

(2) \Rightarrow (1). We can get a Hilbert \mathcal{H} space from $(\mathcal{P}, \langle \cdot, \cdot \rangle)$ after dividing out the kernel $\mathcal{K} = \{P \in \mathcal{P} : \langle P, P \rangle = 0\}$, and completion. From now on, we will identify elements in \mathcal{P} with their images in \mathcal{H} . We adapt the construction in Section 4.7 in [RS] (see also [GSS]). Let $\widehat{\mathcal{H}} = L^2(\mathbb{R}_+) \otimes \mathcal{H}$, and $\mathcal{F}(\widehat{\mathcal{H}}) = \mathbb{C}\Omega \oplus \widehat{\mathcal{H}} \otimes \widehat{\mathcal{H}} \oplus \cdots \oplus \widehat{\mathcal{H}}^{\otimes n} \oplus \cdots$, where Ω is called the vacuum vector of the full Fock space $\mathcal{F}(\widehat{\mathcal{H}})$. Define $\tau : B(\mathcal{F}(\widehat{\mathcal{H}})) \rightarrow \mathbb{C}$, $\tau(T) = \langle T\Omega, \Omega \rangle$, for $T \in B(\mathcal{F}(\widehat{\mathcal{H}}))$. Then $(B(\mathcal{F}(\widehat{\mathcal{H}})), \tau)$ is a C^* -probability space.

For $x \in \widehat{\mathcal{H}}$, and $T \in B(\widehat{\mathcal{H}})$, define $l^*(x)$, $l(x)$ and $p(T)$ on $\mathcal{F}(\widehat{\mathcal{H}})$ as follows.

$$l^*(x)\Omega = x, l^*(x)\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n = x \otimes \xi_1 \otimes \cdots \otimes \xi_n, l(x)\Omega = 0,$$

$$l(x)\xi_1 \otimes \cdots \otimes \xi_n = \langle \xi_1, x \rangle \xi_2 \otimes \cdots \otimes \xi_n, p(T)\Omega = 0, p(T)\xi_1 \otimes \cdots \otimes \xi_n = T\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n,$$

for $\xi_1, \xi_2, \dots, \xi_n \in \widehat{\mathcal{H}}$.

Let $a_0^{(i)} = 0$, and

$$a_t^{(i)} = t\varphi(a_i) + l(\chi_{(0,t)} \otimes X_i) + l^*(\chi_{(0,t)} \otimes X_i) + p(\lambda(\chi_{(0,t)}) \otimes \lambda(X_i)) \in B(\mathcal{F}(\widehat{\mathcal{H}})),$$

for $t > 0$, where $\lambda(\chi_{(0,t)})$ and $\lambda(X_i)$ be the left multiplication operators of $\chi_{(0,t)}$ and X_i on $L^2(\mathbb{R}_+)$ and \mathcal{H} , respectively. Let

$$a_{s,t}^{(i)} = a_t^{(i)} - a_s^{(i)} = (t-s)\varphi(a_i) + l(\chi_{(s,t)} \otimes X_i) + l^*(\chi_{(s,t)} \otimes X_i) + p(\lambda(\chi_{(s,t)}) \otimes \lambda(X_i)),$$

for $0 \leq s < t$. For $0 \leq s < t$ and $0 \leq u < v$, we have $L^2((s,t)) \otimes \mathcal{H}$ is orthogonal to $L^2((u,v)) \otimes \mathcal{H}$, as subspaces of $\widehat{\mathcal{H}}$, if $(s,t) \cap (u,v) = \emptyset$. It follows from Theorem 4.6.15 in [RS] that $\{a_{s,t}^{(i)} : i = 1, 2, \dots\}$ and $\{a_{u,v}^{(i)} : i = 1, 2, \dots\}$ are free.

Let $b_1^{(i)} = (t-s)\varphi(a_i)$, and $b_2^{(i)} = l(\chi_{(s,t)} \otimes X_i) + l^*(\chi_{(s,t)} \otimes X_i) + p(\lambda(\chi_{(s,t)}) \otimes \lambda(X_i))$. Then

$$\begin{aligned} \tau(a_{s,t}^{(i(1))} a_{s,t}^{(i(2))} \cdots a_{s,t}^{(i(n))}) &= \tau\left(\sum_{j_1, j_2, \dots, j_n=1}^2 b_{j_1}^{(i(1))} \cdots b_{j_n}^{(i(n))}\right) \\ &= \sum_{V_1 \cup V_2 = \{1, 2, \dots, n\}, V_1 \cap V_2 = \emptyset} \prod_{k \in V_1} b_1^{(i(k))} \tau(b_2^{(i(k_1))} b_2^{(I(k_2))} \cdots b_2^{(i(k_l))}) \\ &= \sum_{V_1 \cup V_2 = \{1, 2, \dots, n\}, V_1 \cap V_2 = \emptyset} (t-s)^{|V_1|} \prod_{k \in V_1} \varphi(a_{i(k)}) \tau(b_2^{(i(k_1))} b_2^{(I(k_2))} \cdots b_2^{(i(k_l))}), \end{aligned}$$

where $k_1 < k_2 < \cdots < k_l$, $V_2 = \{k_1, k_2, \dots, k_l\}$. Now we show that $\tau(b_2^{(i(1))} \cdots b_2^{(i(n))})$ is a multiple of a power of $t-s$. Note that $b_2^{(i)} = l(\chi_{(s,t)} \otimes X_i) + l^*(\chi_{(s,t)} \otimes X_i) + p(\lambda(\chi_{(s,t)}) \otimes \lambda(X_i))$. Let's use the following notations.

$$b_{2,1} = l(\chi_{(s,t)}), b_{2,2}^{(i)} = l(X_i), b_{3,1} = l^*(\chi_{(s,t)}), b_{3,2}^{(i)} = l^*(X_i), b_{4,1} = p(\lambda(\chi_{(s,t)})), b_{4,2}^{(i)} = p(\lambda(X_i)).$$

Then

$$\tau(b_2^{(i(1))} \cdots b_2^{(i(n))}) = \sum_{j_1, j_2, \dots, j_n=2}^4 \tau\left(\prod_{l=1}^n b_{j_l,1} \otimes b_{j_l,2}^{(i(l))}\right) = \sum_{j_1, j_2, \dots, j_n=2}^4 \tau_1\left(\prod_{l=1}^n b_{j_l,1}\right) \tau_2\left(\prod_{l=1}^n b_{j_l,2}^{(i(l))}\right),$$

where τ_1 and τ_2 are vacuum states on the full Fock spaces $\mathcal{F}(L^2(\mathbb{R}_+))$ and $\mathcal{F}(\mathcal{H})$, respectively. Note that $\tau_1(\prod_{l=1}^n b_{j_l,1}) \neq 0$ implies that $j_n = 3, j_1 = 2$. Therefore,

$$\tau_1(\prod_{l=1}^n b_{j_l,1}) = \langle b_{j_2,1} \cdots b_{j_{n-1},1} \chi_{(s,t)}, \chi_{(s,t)} \rangle.$$

Also, $b_{4,1} \chi_{(s,t)} = \chi_{(s,t)}^2 = \chi_{(s,t)}$. Therefore, we can assume that there is no $b_{4,1}$ in $\{b_{j_2,1}, \dots, b_{j_{n-1},1}\}$. Since $\tau_1(\prod_{l=1}^n b_{j_l,1}) = \langle b_{j_2,1} \cdots b_{j_{n-1},1} \chi_{(s,t)}, \chi_{(s,t)} \rangle \neq 0$, the numbers of $b_{2,1}$'s must be equal to the number of $b_{3,1}$'s in $\{b_{j_2,1}, \dots, b_{j_{n-1},1}\}$. Note also that $b_{2,1} b_{3,1} \xi = (t-s)\xi$, and $b_{3,1} b_{2,1} \chi_{(s,t)} \otimes \xi = (t-s)\chi_{(s,t)} \otimes \xi$, for all $\xi \in \mathcal{F}(L^2(\mathbb{R}_+))$. Therefore, $\tau_1(\prod_{l=1}^n b_{j_l,1})$ is a multiple of a power of $t-s$, if $\tau_1(\prod_{l=1}^n b_{j_l,1}) \neq 0$. It follows that $\tau(b_2^{(i(1))} \cdots b_2^{(i(n))})$ is a multiple of a power of $t-s$. Hence, the distribution of $\{a_{s,t}^{(i)} : i \in \mathbb{N}\}$ is same as that of $\{a_{0,t-s}^{(i)} : i \in \mathbb{N}\}$. The above argument also shows that $\lim_{t-s \rightarrow 0} \tau(a_{s,t}^{(i(1))} a_{s,t}^{(i(2))} \cdots a_{s,t}^{(i(n))}) = 0$, that is, $\{a_t^{(i)} : i = 1, 2, \dots\}$ converges to 0 in distribution, as $t \rightarrow 0$. Hence, $\{\{a_t^{(i)} : i = 1, 2, \dots\} : t \geq 0\}$ is a multidimensional free Levy process.

Now we show that $a_1 := \{a_1^{(i)} : i = 1, 2, \dots\}$ has a distribution same as that of $\{a_n : n = 1, 2, \dots\}$. Since $\{a_t := \{a_t^{(i)} : i \in \mathbb{N}\} : t \geq 0\}$ is a multidimensional free Levy process, $\{a_1^{(i)} : i \in \mathbb{N}\}$ has a multidimensional free infinitely divisible distribution μ_{a_1} . Let $\mu_t = \mu(a_t)$, the distributions of a_t , for $t \geq 0$. Then $\mu_0 = \mu(a_0) = 0$, μ_t converges to zero pointwisely, as $t \rightarrow 0$. Moreover,

$$\mu_{t+s} = \mu(a_{t+s}) = \mu(a_s + a_{s+t} - a_s) = \mu(a_s) \boxplus \mu(a_{s+t} - a_s) = \mu_s \boxplus \mu_t,$$

for $t, s > 0$. Therefore, $\{\mu_t : t \geq 0\}$ is the semigroup of distributions corresponding to μ_{a_1} (see 4.5.3 in [RS]). By Proposition 4.5.4 in [RS], $\kappa_n(a_1^{(i(1))}, \dots, a_1^{(i(n))}) = \lim_{t \rightarrow 0} \frac{1}{t} \tau(a_t^{(i(1))}, \dots, a_t^{(i(n))})$. By the last part of the proof of 4.7.1 in [RS], we get

$$\kappa_n(a_1^{(i(1))}, \dots, a_1^{(i(n))}) = \kappa_n(a_{i(1)}, \dots, a_{i(n)}).$$

(2) \Rightarrow (5). It is obvious from (2) that $\kappa(PP^*) = \langle P, P \rangle \geq 0$, for $P \in \mathcal{P}$. For

$$X_{\underline{i}} = X_{i(1)} X_{i(2)} \cdots X_{i(m)}, X_{\underline{j}} = X_{j(1)} X_{j(2)} \cdots X_{j(n)} \in \mathcal{P},$$

we divide every partition π in $NC_{i,j}$ of all non-crossing partitions of the set $\{i(1), \dots, i(m), j(1), \dots, j(n)\}$ into three parts, $\pi = \pi_i \cup \pi_j \cup \pi_{i,j}$, where π_i consists of subsets of $\{i(1), \dots, i(m)\}$, π_j consists of subsets of $\{j(1), \dots, j(n)\}$, and each block V of $\pi_{i,j}$ contains both i 's and j 's. We then have

$$\begin{aligned} & \kappa(X_{\underline{i}} X_{\underline{j}}) \\ &= \kappa_{m+n}(a_{\underline{i}}, a_{\underline{j}}) = \sum_{\pi \in NC_{i,j}} \varphi_{\pi_i}(a_{i(1)}, \dots, a_{i(m)}) \varphi_{\pi_j}(a_{j(1)}, \dots, a_{j(n)}) \\ & \quad \times \prod_{V=\{i_1, \dots, i_k, j_1, \dots, j_l\} \in \pi_{i,j}} \varphi(a_{i_1} \cdots a_{i_k} a_{j_1} \cdots a_{j_l}) \mu(\pi, 1_{m+n}) \\ &= \sum_{\pi \in NC_{i,j}} \varphi_{\pi_j}(a_{j(1)}, \dots, a_{j(n)}) \varphi_{\pi_i}(a_{i(1)}, \dots, a_{i(m)}) \\ & \quad \times \prod_{V=\{j_1, \dots, j_l, i_1, \dots, i_k\} \in \pi_{j,i}} \varphi(a_{j_1} \cdots a_{j_l} a_{i_1} \cdots a_{i_k}) \mu(\pi, 1_{m+n}) \\ &= \sum_{\pi \in NC_{j,i}} \varphi_{\pi}(a_{j(1)}, \dots, a_{j(n)}, a_{i(1)}, \dots, a_{i(m)}) \mu(\pi, 1_{m+n}) \\ &= \kappa_{m+n}(a_{j(1)}, \dots, a_{j(n)}, a_{i(1)}, \dots, a_{i(m)}) = \kappa(X_{\underline{j}} X_{\underline{i}}), \end{aligned}$$

where $NC_{j,i}$ is the set of all non-crossing partitions of $\{j(1), \dots, j(n), i(1), \dots, i(m)\}$,

$$\pi = \pi_j \cup \pi_i \cup \pi_{j,i}$$

is a partition of $\pi \in NC_{j,i}$, similar to the partition $\pi = \pi_i \cup \pi_j \cup \pi_{i,j}$ of a partition π in $NC_{i,j}$. It follows that $\kappa(PQ) = \kappa(QP)$, for all $p, Q \in \mathcal{P}$. Moreover,

$$\begin{aligned}\kappa(X_{\underline{i}}) &= \kappa_m(a_{i(1)}, a_{i(2)}, \dots, a_{i(m)}) = \langle X_{i(1)} \cdots X_{i(m-1)}, X_{i(m)} \rangle \\ &= \overline{\langle X_{i(m)}, X_{i(1)} \cdots X_{i(m-1)} \rangle} = \overline{\kappa_m(a_{i(m)}, a_{i(m-1)}, \dots, a_{i(1)})} \\ &= \overline{\kappa(X_{\underline{i}_r})}.\end{aligned}$$

Therefore,

$$\kappa(P^*) = \sum \overline{\alpha_{i(1), \dots, i(n)}} \kappa(X_{\underline{i}_r}) = \overline{\kappa(P)}.$$

(5) \Rightarrow (2) is obvious. Define $\langle P, Q \rangle = \kappa(PQ^*)$ for $P, Q \in \mathcal{P}$. Then, by (5), $\langle \cdot, \cdot \rangle$ is non-negative sesquilinear on \mathcal{P} . \square

By our multidimensional compound free Poisson limit theorem (Theorem 2.8), and above Theorem 3.3, we get the following corollary.

Corollary 3.4. *If a sequence $\{b_n : n \in \mathbb{N}\}$ of self-adjoint operators in a C^* -probability space (\mathcal{A}, φ) has a multidimensional compound free Poisson distribution, then its distribution is multidimensional free infinitely divisible.*

By (2) in Theorem 3.3, we have the following result.

Corollary 3.5. *Let $\{\{a_{m,n} : n = 1, 2, \dots\} : m = 1, 2, \dots\}$ be a sequence of sequences of random variables in a C^* -probability space (\mathcal{A}, φ) . If each sequence $\{a_{n,m} : n = 1, 2, \dots\}$ has a multidimensional free infinitely divisible distribution, for $m = 1, 2, \dots$, and $\{\{a_{m,n} : n = 1, 2, \dots\} : m = 1, 2, \dots\}$ converges in distribution to $\{b_n\}$ of random variables in a non-commutative probability space (\mathcal{B}, ϕ) , as $m \rightarrow \infty$, then $\{b_n : n = 1, 2, \dots\}$ has a multidimensional free infinitely divisible distribution.*

The following theorem generalizes Speicher's free compound Poisson distribution approximation theorem 4.5.5 in [RS] to the case of infinitely dimensional distributions.

Theorem 3.6. *A sequence $\{b_n : n = 1, 2, \dots\}$ of self-adjoint operators in a C^* -probability space (\mathcal{A}, φ) has a multidimensional free infinitely dimensional distribution if and only if there is a bi-index sequence $\{\{p_{m,n} : n = 1, 2, \dots\} : m = 1, 2, \dots\}$ of self-adjoint operators in a C^* -probability space (\mathcal{B}, ϕ) such that $\{p_{m,n} : n = 1, 2, \dots\}$, for $m = 1, 2, \dots$, have multidimensional compound free Poisson random distributions and $\{p_{m,n} : n = 1, 2, \dots\}$ converges in distribution to $\{b_n : n = 1, 2, \dots\}$, as $m \rightarrow \infty$.*

Proof. By the above two corollaries, it is sufficient to show that every sequence of random variables having a multidimensional free infinitely divisible is the limit in distribution of a sequence of sequences of random variables having multidimensional compound free Poisson distributions. Let (\mathcal{A}, φ) be a C^* -probability space and $\{b_n : n = 1, 2, \dots\}$ be a sequence of self-adjoint operators in \mathcal{A} having a multidimensional free infinitely divisible distribution. Then for each $m \in \mathbb{N}$, $\{b_1, \dots, b_m\}$ has a multidimensional free infinitely divisible distribution. By 4.5.5 in [RS], there is a sequence $\{\{p_{i,m}^{(j)} : i = 1, 2, \dots, m, j = 1, 2, \dots\}\}$ of random variables such that $\{p_{1,m}^{(j)}, \dots, p_{m,m}^{(j)}\}$ has a multidimensional compound free Poisson distribution, for $j = 1, 2, \dots$, and its distribution converges to that of $\{b_1, \dots, b_m\}$, as $j \rightarrow \infty$. By Theorem 2.8, we can choose $p_{1,m}^{(j)}, \dots, p_{m,m}^{(j)}$ from a C^* -probability space $(\mathcal{B}_{m,j}, \varphi_{m,j})$. Without loss of generality, we can assume $\mathcal{B}_{m,j}$ is generated by $\{p_{i,m}^{(j)} : i = 1, 2, \dots, m\}$. Define a homomorphism $f_{m,j} : \mathcal{B}_{m,j} \rightarrow \mathcal{B}_{m+1,j}$, for $m \geq 1, j = 1, 2, \dots$, by $f_{m,j}(p_{i,m}^{(j)}) = p_{i,m+1}^{(j)}$, for $i = 1, 2, \dots, m$. For $m < n$, define $f_{m,n,j} = f_{n-1,j} \circ \dots \circ f_{m+1,j} \circ f_{m,j}$. Then $\{(\mathcal{B}_{m,j}, f_{m,j}) : m = 1, 2, \dots\}$ is a directed system. Moreover, by the proof of 4.5.5 in [RS], $\{p_{1,m}^{(j)}, \dots, p_{m,m}^{(j)}\}$ and $\{f_{m,j}(p_{i,m}^{(j)}) : i = 1, 2, \dots, m\}$ have the same distribution, for $m, j = 1, 2, \dots$. We define an equivalent relation \equiv in the disjoint union $\bigsqcup_{m=1}^{\infty} \mathcal{B}_{m,j}$ as follows. For $a \in \mathcal{B}_{m,j}, b \in \mathcal{B}_{n,j}$,

we say $a \equiv b$ if there is a $k > m, k > n$ such that $f_{m,k,j}(a) = f_{n,k,j}(b)$. For $a \in \bigsqcup_m \mathcal{B}_{m,j}$, we define $[a]$ to be the equivalent class of a in $\bigsqcup_m \mathcal{B}_{m,j} / \equiv$. Define a semi-norm $\|\cdot\|$ on $\bigsqcup_m \mathcal{B}_{m,j} / \equiv$ as follows.

$$\|[a]\| = \lim_{n \rightarrow \infty} \|f_{m,n,j}(a)\|,$$

for $a \in \mathcal{B}_{m,j}$. We can get a C^* -algebra \mathcal{B}_j from $(\bigsqcup_m \mathcal{B}_{m,j} / \equiv, \|\cdot\|)$ after dividing out the kernel $I = \{[x] \in \bigsqcup_m \mathcal{B}_{m,j} / \equiv : \|[x]\| = 0\}$ and completion, which is called the directed limit C^* -algebra of $(\mathcal{B}_{m,j}, f_{m,j})$ (see Pages 38-41 in [KT]). Define a linear functional $\varphi_j : \mathcal{B}_j \rightarrow \mathbb{C}$, $\varphi_j([a]) = \varphi_{m,j}(a)$, for $a \in \mathcal{B}_{m,j}$. Then $(\mathcal{B}_j, \varphi_j)$ is a C^* -probability space. Let $[p_i^{(j)}] = [p_{i,m}^{(j)}]$, for $m \geq i, i, j = 1, 2, \dots$. For $1 \leq i(1), i(2), \dots, i(n) \leq m$, we have

$$\varphi_j([p_{i(1)}^{(j)}] \cdots [p_{i(n)}^{(j)}]) = \varphi_{m,j}(p_{i(1),m}^{(j)} \cdots p_{i(n),m}^{(j)}).$$

It follows that $\{[p_i^{(j)}] : i = 1, 2, \dots\}$ has a multidimensional compound free Poisson distribution, for $j = 1, 2, \dots$. Moreover,

$$\varphi(b_{i(1)} \cdots b_{i(n)}) = \lim_{j \rightarrow \infty} \varphi_{m,j}(p_{i(1),m}^{(j)} \cdots p_{i(n),m}^{(j)}) = \lim_{j \rightarrow \infty} \varphi_j([p_{i(1)}^{(j)}] \cdots [p_{i(n)}^{(j)}]).$$

The conclusion follows now. □

REFERENCES

- [BnT] O. E. Barndorff-Nielsen and S. Thorjornsen. *Classical and free infinite divisibility and Levy processes*. Lecture Note in Math. 1866 (2006), 33-160. Springer-Verlag Berlin-Heidelberg.
- [BG] F. Benauch-Georges. *Multidimensional free infinitely divisible distributions*. <http://www.cmapx.polytechnique.fr/benaych/from.Ph.D.01.06.PDF>, 2006.
- [BP] H. Bercovici and V. Pata. *Stable laws and domains of attraction in free probability theory*. Ann. of Math., 149(1999), 1023-1060.
- [WF] W. Feller. *An Introduction to Probability Theory and its Applications*, Vol. II, 2nd edition, Hohn Wiley and Sons, Inc., 1970.
- [MG] M. Gao. *Two-faced Families of Non-commutative Random Variables Having Bi-free Infinitely Divisible Distributions*. Internat. J. Math., DOI: 10.1142/S0129167X16500373, April, 2016.
- [GH] L. Ge and D. Hadwin. *Ultraproducts of C^* -algebras*. Operator Theory: Advances and Applications, Vol. 127, 305-326. Beikhauser Verlag, Basel, Switzerland, 2001.
- [GSS] P. Glockner, M. Schurmann and R. Speicher. *Realization of free white noises*. Arch. Math., Vol58(1992), 407-416.
- [GHM] Y. Gu, H. Huang, and J. Mingo. *An analogue of the Levy-Hinchin formula for bi-free infinitely divisible distributions*. arXiv:1501.05369v2 [math.OA], 9 July 2015.
- [KR] R. Kadison and J. Ringrose. *Fundamentals of the theory of operator algebras*. Graduate Studies in Math. Vol. 16, AMS, 1997.
- [AK] A. Kruckman. *Notes on Ultrafilters*. <https://math.berkeley.edu/~kruckman/ultrafilters.pdf>.
- [NS] A. Nica and R. Speicher. *Lectures on Combinatorics for Free Probability*, LMS Lecture Notes 335, Cambridge University Press, 2006.
- [SS] A. Sinclair and R. Smith. *Finite von Neumann algebras and Masas*. London Math. Soc. Lecture Notes Series 351. Cambridge Univer. Press, 2008.
- [RS] R. Speicher. *Combinatorial theory for the free product with amalgamation and operator-valued free probability theory*. Mem. Amer. Math. soc., 132(1998), no. 627.
- [RS1] R. Speicher. *A new example independence and white noise*. Probability Theory and Related Fields 84(1990), 141-159.
- [KT] K. Thomsen. *Classifying C^* -algebras*. <http://home.math.au.dk/matkt/BOG.pdf>.
- [DV] D. Voiculescu. *Summetries of some reduced free product C^* -algebras*. Lecture Notes in Math., Vol. 1132, Springer Verlag, 1985, 556-588.
- [DV1] D. Voiculescu. *Limit Laws for random matrices and free products*. Invent. Math., 104(1991), 201-220.
- [DV2] D. Voiculescu. *Free probability for pairs of faces*. Comm. Math. Phys., 332(2014), No.3, 955-980.
- [VDN] D. Voiculescu, K. Dykema, and A. Nica. *Free Random variables*. CRM Monograph Series, Vol. 1, AMS, 1992.

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